Photon adding and subtracting and Schrödinger-cat generation in conditional output measurement on a beam splitter

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Abstract

The problem of photon adding and subtracting is studied, using conditional output measurement on a beam splitter. It is shown that for various classes of states the corresponding photon-added and -subtracted states can be prepared. Analytical results are presented, with special emphasis on photon-added and -subtracted squeezed vacuum states, which are found to represent two different types of Schrödinger-cat-like states. Effects of realistic photocounting and Fock-state preparation are discussed.

1 Introduction

Quantum state engineering has been a subject of increasing interest and various methods for generating highly nonclassical states of optical fields have been proposed. A promising method for generating novel quantum states has been conditional measurement. In fact, when a system, such as a correlated two-mode optical field or a correlated atom-field system, is prepared in an entangled state of two subsystems and a measurement is performed on one subsystem, then the quantum state of the other subsystem can be reduced to a new state. In particular, it turned out that conditional measurement on a beam splitter may be advantageously used for generating new classes of quantum states [1].

Very interesting states are the so-called photon-added and photon-subtracted states that are obtained by repeated application of the photon creation and destruction operator, respectively, on a given state [1, 2, 3, 4]. In this paper we show that for various classes of states the corresponding photon-added and -subtracted states can be produced by means of conditional measurement on a beam splitter. In particular, when a signal mode is mixed with a reference mode prepared in a photon-number state and in one of the output channels no photons are detected, then the mode in the other output channel is prepared in a photon-added state. Accordingly, a photon-subtracted state is produced when the reference mode is in the vacuum and the conditional output measurement yields a nonvanishing number of photons. Typical examples of states the method applies to are thermal states, coherent states, squeezed states, displaced photon-number states and others. It is worth noting that photon-added and photon-subtracted squeezed vacuum states represent two different classes of Schrödinger-cat-like states. We analyze the states in terms of quadrature-component distributions and phase-space functions and calculate the probability of producing them. Finally we discuss modifications that may be observed under realistic experimental conditions.

2 Generation of photon-added and -subtracted states

It is well known that the input–output relations at a lossless beam splitter can be treated within the SU(2) algebra. In the Schrödinger picture, the output-density operator $\hat{\varrho}_{\rm out}$ can be obtained from the input–density operator $\hat{\varrho}_{\rm in}$ as $\hat{\varrho}_{\rm out} = \hat{V}^{\dagger}\hat{\varrho}_{\rm in}\hat{V}$, where the operator $\hat{V}$ is given by $\hat{V} = e^{-i(\varphi_T-\varphi_R)\hat{L}_3}\, e^{-2i\theta\hat{L}_2}\, e^{-i(\varphi_T+\varphi_R)\hat{L}_3}$, with $\hat{L}_2 = \frac{1}{2i}(\hat{a}_1^{\dagger}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{a}_1)$ and $\hat{L}_3 = \frac{1}{2}(\hat{a}_1^{\dagger}\hat{a}_1 - \hat{a}_2^{\dagger}\hat{a}_2)$. Let us assume that the input-state density operator can be written as

$$\hat{\varrho}_{\rm in}(n_0) = \hat{\varrho}_{\rm in1} \otimes |n_0\rangle\langle\, n_0|, \quad n_0 = 0, 1, 2... \tag{1}$$

($\hat{\varrho}_{\rm in1}$, density operator of the mode in the first input channel; $|n_0\rangle$, Fock state of the second input mode), then the output-state density operator $\hat{\varrho}_{\rm out} \equiv \hat{\varrho}_{\rm out}(n_0)$ can be given by

$$\begin{aligned}\hat{\varrho}_{\rm out}(n_0) = {}& \frac{1}{|T|^{2n_0}} \sum_{n_2=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{k=0}^{n_0}\sum_{j=0}^{n_0} (R^*)^{m_2+j} R^{n_2+k} \\ & \times \frac{(-1)^{n_2+m_2}}{\sqrt{k!j!m_2!n_2!}} \sqrt{\binom{n_0}{k}\binom{n_0}{j}\binom{n_0-k+m_2}{m_2}\binom{n_0-j+n_2}{n_2}} \\ & \times T^{\hat{n}_1}\hat{a}_1^{m_2}(\hat{a}_1^{\dagger})^k\hat{\varrho}_{\rm in1}\hat{a}_1^{j}(\hat{a}_1^{\dagger})^{n_2}(T^*)^{\hat{n}_1} \otimes |n_0-k+m_2\rangle\langle n_0-j+n_2|, \end{aligned} \tag{2}$$

where $T = \cos\theta\; e^{i\varphi_T}$ and $R = \sin\theta\; e^{i\varphi_R}$, respectively, are the reflectance and transmittance of the beam splitter. From Eq. (2) we see that the output modes are, in general, highly correlated. When the photon number of the mode in the second output channel is measured and m_2 photons are detected, then the mode in the first output channel is prepared in a quantum state whose density operator $\hat{\varrho}_{\rm out1}(n_0, m_2)$ reads as

$$\hat{\varrho}_{\rm out1}(n_0, m_2) = \frac{\langle m_2|\hat{\varrho}_{\rm out}(n_0)|m_2\rangle}{{\rm Tr}_1(\langle m_2|\hat{\varrho}_{\rm out}(n_0)|m_2\rangle)}\,. \tag{3}$$

The probability of such an event is given by

$$\begin{aligned}P(n_0, m_2) = {\rm Tr}_1(\langle m_2|\hat{\varrho}_{\rm out}(n_0)|m_2\rangle) = {}& \sum_{n_1=\mu-\nu}^{\infty}\sum_{j=\mu}^{n_0}\sum_{k=\mu}^{n_0} |R|^{2(j+k-\nu)}|T|^{2(n_1+\nu-n_0)} \\ & \times \frac{(-1)^{j+k}n_0!\,n_1!}{(n_1+\nu)!(n_0-\nu)!}\binom{n_0-\nu}{j-\nu}\binom{n_0-\nu}{k-\nu}\binom{n_1+j}{j}\binom{n_1+k}{k}\langle n_1|\hat{\varrho}_{\rm in1}|n_1\rangle, \end{aligned} \tag{4}$$

where $\nu = n_0 - m_2$, $\mu = \max(0, \nu)$.

From Eqs. (2) and (3) we find that when a signal mode prepared in a state $\hat{\varrho}_{\rm in1} = \sum_\Phi \tilde{p}_\Phi\, |\Phi\rangle\langle\Phi|$ ($\sum_\Phi \tilde{p}_\Phi = 1$, $0 \leq \tilde{p}_\Phi \leq 1$) is mixed with a mode prepared in a Fock state and a zero-photon conditional measurement is performed in the second output channel ($n_0 > 0$, $m_2 = 0$), then the mode in the first output channel is prepared in the state

$$\hat{\varrho}_{\rm out1}(n_0 \neq 0, m_2 = 0) = \sum_\Phi \tilde{p}_\Phi |\Psi_{n_0,0}\rangle\langle\Psi_{n_0,0}|, \quad |\Psi_{n_0,0}\rangle = \mathcal{N}_{n_0,0}^{-1/2}\,(\hat{a}_1^{\dagger})^{n_0}\,T^{\hat{n}_1}|\Phi\rangle. \tag{5}$$

As can be seen, the state $|\Psi_{n_0,0}\rangle$ is a photon-added state, n_0 photons beeing added to the state $T^{\hat{n}_1}|\Phi\rangle$. For a number of classes of states (e.g., thermal, coherent, squeezed, displaced Fock states etc.) the states $|\Phi\rangle$ and $T^{\hat{n}_1}|\Phi\rangle$ belong to the same class.In this case conditional output measurement provides us with a method for generating photon-added states of the class of states to which a signal-mode quantum state $|\Phi\rangle$ belongs. Similarly, when the second input mode is in the vacuum state and a nonvanishing number of photons is detected ($n_0 = 0$, $m_2 > 0$), then a photon-subtracted state is produced:

$$\hat{\varrho}_{\rm out1}(n_0 = 0, m_2 \neq 0) = \sum_\Phi \tilde{p}_\Phi |\,\Psi_{0,m_2}\rangle\langle\Psi_{0,m_2}|, \quad |\Psi_{0,m_2}\rangle = \mathcal{N}_{0,m_2}^{-1/2}\,(\hat{a}_1)^{m_2}\,T^{\hat{n}_1}|\Phi\rangle. \tag{6}$$

In Eqs. (5) and (6), $\mathcal{N}_{0,m_2}$ and $\mathcal{N}_{n_0,0}$ are normalization constants. The probabilities of observing the photon-added and photon-subtracted states, respectively, are found from Eq. (4) to be

$$P(n_0) \equiv P(n_0 \neq 0, m_2 = 0) = |R|^{2n_0} \sum_{n_1=0}^{\infty} |T|^{2n_1} \binom{n_1+n_0}{n_0} \langle n_1|\hat{\varrho}_{\text{in}1}|n_1\rangle \tag{7}$$

and

$$P(m_2) \equiv P(n_0 = 0, m_2 \neq 0) = |R|^{2m_2} \sum_{n_1=0}^{\infty} |T|^{2n_1} \binom{n_1+m_2}{m_2} \langle n_1+m_2|\hat{\varrho}_{\text{in}1}|n_1+m_2\rangle. \tag{8}$$

In order to illustrate the method, let us restrict attention to photon-added and photon-subtracted squeezed-vacuum states.

3 Photon-added and -subtracted squeezed vacuum states

When the input state $|\Phi\rangle$ is a squeezed vacuum state,

$$|\Phi\rangle = |0\rangle_{\text{s}} = (1-|\kappa|^2)^{1/4} \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^n\, n!}\, \kappa^n |2n\rangle, \tag{9}$$

then the conditional output states $|\Psi_{n_0,0}\rangle$, Eq. (5), and $|\Psi_{0,m}\rangle$, Eq. (6), are given by

$$|\Psi_{n_0,0}\rangle = \mathcal{N}_{n_0,0}^{-1/2} \sum_{n=n_0}^{\infty} c_{n,n_0,0}|n\rangle, \quad c_{n,n_0,0} = \frac{\sqrt{n!}}{\Gamma\Big[\frac{1}{2}(n-n_0)+1\Big]}\, \tfrac{1}{2}\left[1+(-1)^{n-n_0}\right] \left(\tfrac{1}{2}\kappa'\right)^{\frac{n-n_0}{2}}, \tag{10}$$

and

$$|\Psi_{0,m}\rangle = \mathcal{N}_{0,m}^{-1/2} \sum_{n=0}^{\infty} c_{n,0,m}|n\rangle, \quad c_{n,0,m} = \frac{(m+n)!}{\Gamma\Big[\frac{1}{2}(n+m)+1\Big]\sqrt{n!}}\, \tfrac{1}{2}\left[1+(-1)^{n+m}\right] \left(\tfrac{1}{2}\kappa'\right)^{\frac{n+m}{2}}, \tag{11}$$

where $\kappa' = T^2\kappa$, and the normalization constants are derived to be

$$\mathcal{N}_{0,m} = \frac{|\kappa'|^{2m}}{(1-|\kappa'|^2)^{m+1/2}} \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(m!)^2\, (2|\kappa'|)^{-2k}}{(m-2k)!\, (k!)^2}, \quad \mathcal{N}_{n_0,0} = n_0!\, \mathrm{F}\Big[\tfrac{1}{2}(n_0+1), \tfrac{1}{2}(n_0+2), 1; |\kappa'|^2\Big]. \tag{12}$$

For notational convenience we omit the subscripts 1 and 2 introduced above to distinguish between the two output channels. The probabilities of producing the states $|\Psi_{n_0,0}\rangle$ and $|\Psi_{0,m}\rangle$ are

$$P(n_0) = |R|^{2n_0}\sqrt{1-|\kappa|^2}\, \mathrm{F}\Big(n_0+1, \tfrac{1}{2}, 1; |\kappa'|^2\Big) \tag{13}$$

and

$$P(m) = \frac{2^{-m}|R|^{2m}\sqrt{1-|\kappa|^2}}{(1-|\kappa'|^2)^{m+1/2}|T|^{2m}} \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{m!(2|\kappa'|)^{m-2k}}{(m-2k)!(k!)^2}, \tag{14}$$

respectively [$\mathrm{F}(a,b,c;z)$, hypergeometric function].

The states $|\Psi_{n_0,0}\rangle$ and $|\Psi_{0,m}\rangle$ are Schrödinger-cat-like states [5], i.e., superpositions of two (macroscopically distinguishable) “component” states that are well localized in the phase space. The properties of $|\Psi_{n_0,0}\rangle$ and $|\Psi_{0,m}\rangle$ can be seen from the quadrature-component distributions $p_{n_0}(x,\varphi|0)$ and $p_0(x,\varphi|m)$:

$$p_{n_0}(x,\varphi|0) \;=\; \frac{1}{\mathcal{N}_{n_0,0}\sqrt{\pi\Delta^{n_0+1}}2^{n_0}}\, \exp\!\left(-\frac{1-|\kappa'|^2}{\Delta}\, x^2\right) \left|\mathrm{H}_{n_0}\Big[\sqrt{(1+\kappa'^{*}e^{i2\varphi})/\Delta}\, x\Big]\right|^2, \tag{15}$$

$$p_0(x,\varphi|m) \;=\; \frac{|\kappa'|^m}{\mathcal{N}_{0,m}\sqrt{\pi\Delta^{m+1}}\, 2^m}\, \exp\!\left(-\frac{1-|\kappa'|^2}{\Delta}\, x^2\right) \left|\mathrm{H}_m\Big[\sqrt{(-\kappa'^{*}e^{i2\varphi}-|\kappa'|^2)/\Delta}\, x\Big]\right|^2, \tag{16}$$

where $\varphi_{\kappa'} = \varphi_\kappa + 2\varphi_T$, and $\Delta = 1 + |\kappa'|^2 + 2|\kappa'|\cos(2\varphi - \varphi_{\kappa'})$ [$\mathrm{H}_n(z)$, Hermite polynomial]. The corresponding Wigner functions are calculated to be

$$W_{n_0}(x,p|0) = \frac{|\kappa'|^{n_0}}{\pi\mathcal{N}_{n_0,0}2^{n_0}|1+\kappa'|^{2n_0+1}} \left(\frac{2}{\lambda+\lambda^*}\right)^{n_0+1/2} \exp\!\left(-\frac{2|\lambda|^2}{\lambda+\lambda^*}\left|x+i\frac{p}{\lambda}\right|^2\right)$$
$$\times \sum_{k=0}^{n_0} \binom{n_0}{k}^2 k! \left(\frac{-2}{|\kappa'|}\right)^k \left|\mathrm{H}_{n_0-k}\left[i\sqrt{\frac{2\lambda^2(1+\lambda^*)}{(1-\lambda)(\lambda+\lambda^*)}}\left(x+i\frac{p}{\lambda}\right)\right]\right|^2, \quad (17)$$

$$W_0(x,p|m) = \frac{|\kappa'|^{m}}{\pi\mathcal{N}_{0,m}2^{m}|1+\kappa'|^{2m+1}} \left(\frac{2}{\lambda+\lambda^*}\right)^{m+1/2} \exp\!\left(-\frac{2|\lambda|^2}{\lambda+\lambda^*}\left|x+i\frac{p}{\lambda}\right|^2\right)$$
$$\times \sum_{k=0}^{m} \binom{m}{k}^2 k! \,(-2|\kappa'|)^k \left|\mathrm{H}_{m-k}\left[i\sqrt{\frac{2\lambda^2(1-\lambda^*)}{(1+\lambda)(\lambda+\lambda^*)}}\left(x+i\frac{p}{\lambda}\right)\right]\right|^2, \quad (18)$$

where $\lambda = (1-\kappa')\,/\,(1+\kappa')$, and the Husimi functions read as [$\alpha = 2^{1/2}(x+ip)$]

$$Q_{n_0}(x,p|0) = \frac{|\alpha|^{2n_0}e^{-|\alpha|^2}}{2\pi\mathcal{N}_{n_0,0}} \exp\!\left[\tfrac{1}{2}(\kappa'^*\alpha^2 + \kappa'\alpha^{*2})\right], \quad (19)$$

$$Q_0(x,p|m) = \frac{|\kappa'|^m e^{-|\alpha|^2}}{2\pi\mathcal{N}_{0,m}2^m} \exp\!\left[\tfrac{1}{2}(\kappa'^*\alpha^2 + \kappa'\alpha^{*2})\right] \left|\mathrm{H}_m\left(\sqrt{-\tfrac{1}{2}\kappa'^*}\alpha\right)\right|^2. \quad (20)$$

4 State mixing in real experiments

Let us first address the problem of realistic photon-number measurement in photon-subtracted state generation. Since highly efficient and precisely discriminating photodetectors are not available at present, multichannel photon chopping [6] may be used. For a $2N$-port apparatus the probability of recording k coincident events when m photons are present is given by

$$\tilde{P}_{N,\eta}(k|m) = \sum_l \tilde{P}_N(k|l)\, M_{l,m}(\eta) \quad (21)$$

(η, detection efficiency), where $\tilde{P}_N(k|m) = M_{l,m}(\eta) = 0$ for $k, l > m$, and

$$\tilde{P}_N(k|m) = \frac{1}{N^m}\binom{N}{k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}(k-l)^m \quad \text{for} \quad k \le m, \quad (22)$$

$$M_{l,m}(\eta) = \binom{m}{l}\eta^l(1-\eta)^{m-l} \quad \text{for} \quad l \le m. \quad (23)$$

Since detection of k coincident events can result from various numbers of photons, m, the conditional output state is in general a statistical mixture. In place of $|\Psi_{0,m}\rangle$, Eq. (6), we have

$$\hat{\varrho} = \sum_m P_{N,\eta}(m|k)\, |\Psi_{0,m}\rangle\langle\Psi_{0,m}|, \quad (24)$$

where the conditional probability $P_{N,\eta}(m|k)$ can be obtained as, on using the Bayes rule,

$$P_{N,\eta}(m|k) = \frac{1}{\tilde{P}_{N,\eta}(k)}\tilde{P}_{N,\eta}(k|m)P(m). \quad (25)$$

Here $P(m)$ is the prior probability (14) of m photons being present, and

$$\tilde{P}_{N,\eta}(k) = \sum_m \tilde{P}_{N,\eta}(k|m)P(m) \quad (26)$$

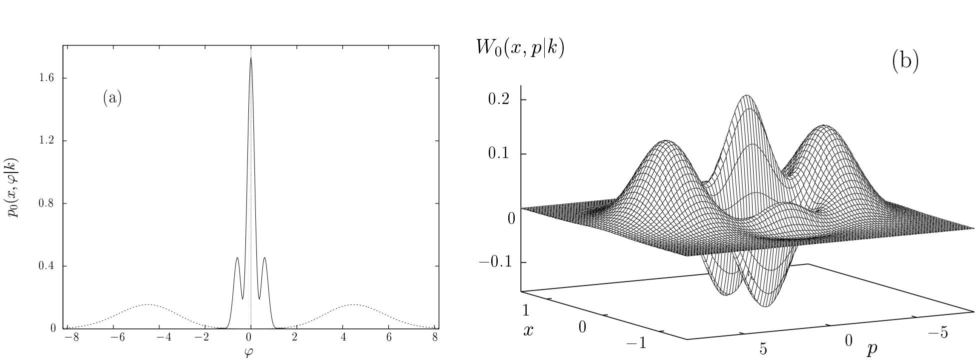

FIG. 1. Quadrature distribution (a) [for the phase parameters $\varphi = 0$ (full line) and $\varphi = \pi/2$ (broken line)] and the Wigner function (b) of a photon-subtracted squeezed vacuum with realistic photodetection ($N = 20$, $\eta = 0.95$) for $k = 4$ coincident events, and $|\kappa| = 0.77$, $|T|^2 = 0.9$ ($\kappa' = -0.7$). $P_{N,\eta}(k) = 0.07\%$

is the prior probability of recording k coincident events. In Figs. 1(a) and 1(b), examples of the resulting quadrature distribution $p_0(x,\varphi|k) = \sum_m P_{N,\eta}(m|k)\, p_0(x,\varphi|m)$ and the Wigner function $W_0(x,p|k) = \sum_m P_{N,\eta}(m|k)\, W_0(x,p|m)$, respectively, are plotted.

In order to produce photon-added states, the second input mode must be prepared in a Fock state. This is a nontrivial experimental task, and a number of proposals have been made. Therefore it may be more realistic to consider a sub-Poissonian statistical mixture of Fock states rather than a pure state. Let us return to Eq. (1) and suppose that $\hat{\varrho}_{\rm in} = \hat{\varrho}_{\rm in1} \otimes \hat{\varrho}_{\rm in2}$, with $\hat{\varrho}_{\rm in2} = \sum_{n_0} \tilde{p}_{n_0} |n_0\rangle\langle n_0|$. To give an example, we assume that $\tilde{p}_{n_0}$ is a binomial probability distribution,

$$\tilde{p}_{n_0} = \binom{N}{n_0} p^{n_0} (1-p)^{N-n_0} \quad \text{if} \quad n_0 \le N, \tag{27}$$

and $\tilde{p}_{n_0} = 0$ elsewhere ($0 < p < 1$). We then find that $|\Psi_{n_0,0}\rangle$, Eq. (5), must be replaced with the mixed state

$$\hat{\varrho} = \sum_{n_0} \tilde{p}_{n_0} \, |\Psi_{n_0,0}\rangle\langle\Psi_{n_0,0}|. \tag{28}$$

Accordingly, the probability of detecting the state is the average of $P(n_0)$, Eq. (13), i.e., $P = \sum_{n_0} \tilde{p}_{n_0} P(n_0)$. Examples of the resulting quadrature-component distribution $p(x,\varphi|0) = \sum_{n_0} \tilde{p}_{n_0}\, p_{n_0}(x,\varphi|0)$ and the Wigner function $W(x,p|0) = \sum_{n_0} \tilde{p}_{n_0}\, W_{n_0}(x,p|0)$, respectively, are plotted in Fig. 2(a) and (b).

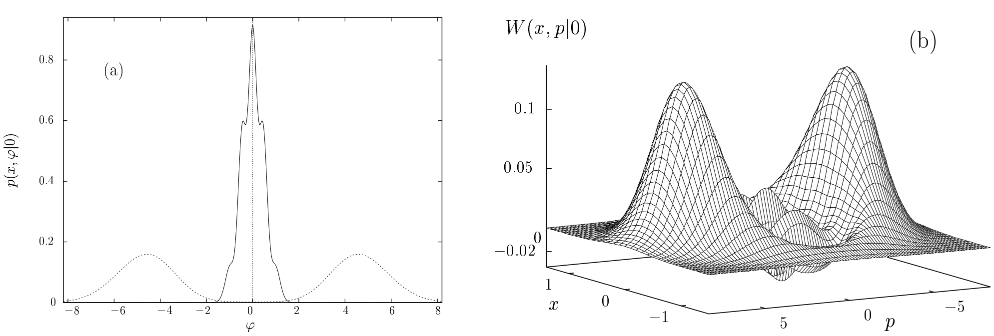

FIG. 2. Quadrature distribution (a) [for the phase parameters $\varphi = 0$ (full line) and $\varphi = \pi/2$ (broken line)] and the Wigner function (b) of a photon-added squeezed vacuum with realistic Fock-state preparation [$p = 0.8$ and $N = 4$ in Eq. (27)] for $|\kappa| = 0.77$, $|T|^2 = 0.9$ ($\kappa' = -0.7$). $P = 0.02\%$

Figures. 1 and 2 reveal that – apart from some smearing – the non-classical Schrödinger-cat-like features of photon-subtracted and photon-added squeezed vacuum states can still be preserved even under realistic experimental conditions. In particular it is seen that the typical quantum interferences can be observed.

5 Summary and Conclusion

We have studied the feasibility of generating photon-added and photon-subtracted states of a single-mode optical field via conditional output measurement on a beam splitter, with special emphasis on photon-added and -subtracted squeezed vacuum states. When a squeezed vacuum and an ordinary vacuum are mixed by a beam splitter and in one of the output channels a nonvanishing number of photons is detected, then a photon-subtracted squeezed vacuum in the other output channel is prepared, which shows all the features of a Schrödinger-cat-like state. Another class of Schrödinger-cat-like states can be prepared when photons are not subtracted from a squeezed vacuum but added to it, which is the case when a squeezed vacuum is mixed with a photon-number state and a zero-photon conditional output measurement is performed. We have derived the quadrature-component distributions and the Wigner and Husimi functions for the two classes of states. Further we have given the probabilities of preparing the states.

It is worth noting that the component states of a Schrödinger-cat-like state of the type of a photon-added squeezed vacuum can be regarded as non-Gaussian squeezed coherent states that tend to the familiar Gaussian squeezed coherent states (two-photon coherent states) as the number of added photons becomes sufficiently large. In contrast to photon adding, the component states of a Schrödinger-cat-like state of the type of a photon-subtracted squeezed vacuum are, in a very good approximation, Gaussian squeezed coherent states which with increasing number of subtracted photons approach ordinary coherent states.

With regard to realistic experimental conditions, we have considered multichannel photon-number detection in photon-subtracted state generation as well as sub-Poissonian photon-number input statistics in photon-added state generation. As expected, realistic photon-number measurement smears the interference structure in the photon-subtracted squeezed vacuum, and a similar effect is observed in the photon-added squeezed vacuum when one allows for an input mode prepared in a sub-Poissonian statistical mixture of photon-number states in place of a pure Fock state. Nevertheless, the typical properties of Schrödinger-cat-like states can still be found even under realistic conditions.

Acknowledgment This work was supported by the Deutsche Forschungsgemeinschaft.

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